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LETTER TO THE EDITOR

On an equivalence of chaotic attractors

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Abstract. In this paper a topological definition of the equivalence between chaotic attractors is analysed by considering two examples in detail.

In spite of the significant progress that has been made in recent years in the understanding of chaos, the general problem of establishing and analysing chaotic attractors of differential equations is still wide open [1]. This letter addresses the question of the equivalence of chaotic attractors. Let us assume that the dynamical systems $\dot{x} = f(x)$ and $\dot{y} = g(y)$ have chaotic attractors A_f and A_g respectively. For simplicity, we will assume that $x = (x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{R}^3$ and $y = (y_1, y_2, y_3)^{\mathrm{T}} \in \mathbb{R}^3$. The following definition ensures topological equivalence of two chaotic attractors: an attractor A_f is equivalent to A_g if there exists a homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(A_f) = A_g$.

Equivalent attractors are said to be of the same type and each equivalence class of chaotic attractors is an attractor type. The above definition is 'strong' in the sense that there is a large number of different attractor types. The question of a 'weaker' definition (and only a small number of attractor types) will be addressed in a forthcoming paper; the main purpose of this letter is to present examples of equivalent chaotic attractors.

Piecewise-linear systems. Consider a class $\mathcal{C}$ of piecewise-linear continuous dynamical systems, defined by a state equation:

$$\dot{x} = f(x) \equiv \begin{cases} \mathbf{A}_1 x + b & x_1 \geqslant 1 \\ \mathbf{A}_0 x & -1 \leqslant x_1 \leqslant 1 \\ \mathbf{A}_{-1} x + c & x_1 \leqslant -1 \end{cases} \tag{1}$$

where $\mathbf{A}_0 = [a_{ij}]$ is a 3×3 matrix, $b = (b_1, b_2, b_3)^{\mathrm{T}}$, $c = (c_1, c_2, c_3)^{\mathrm{T}}$ and

$$\mathbf{A}_1 = \mathbf{A}_0 - \begin{bmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 0 \end{bmatrix} \qquad \mathbf{A}_{-1} = \mathbf{A}_0 + \begin{bmatrix} c_1 & 0 & 0 \\ c_2 & 0 & 0 \\ c_3 & 0 & 0 \end{bmatrix}. \tag{2}$$

Equations (1) and (2) define a 15-parameter family of ordinary differential equations with parameters: $a_{i,j}$, b_i and c_i; $i, j = 1, 2, 3$. Let $(\lambda_1, \lambda_2, \lambda_3)$, (μ_1, μ_2, μ_3) and (ν_1, ν_2, ν_3) denote the eigenvalues of the matrices $\mathbf{A}_0$, $\mathbf{A}_1$ and $\mathbf{A}_{-1}$ respectively. We will henceforth refer to these eigenvalues as the dynamical system's eigenvalues. Define

$$\mathbf{K} = \begin{bmatrix} 1 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \qquad k_{3i} = \sum_{j=1}^{3} a_{1j} a_{ji} \qquad i = 1, 2, 3. \tag{3}$$

Proposition 1. Let C_0 be a subset of C such that $\det \mathbf{K} = 0$. Then if two dynamical systems $f, g \in C \backslash C_0$ have the same eigenvalues, A_f is equivalent to A_g.

Proof. Let us define the following:

$$p_1 = \lambda_1 + \lambda_2 + \lambda_3 \qquad q_1 = \mu_1 + \mu_2 + \mu_3 \qquad r_1 = \nu_1 + \nu_2 + \nu_3$$
$$p_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \qquad q_2 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \qquad r_2 = \nu_1\nu_2 + \nu_1\nu_3 + \nu_2\nu_3$$
$$p_3 = \lambda_1\lambda_2\lambda_3 \qquad q_3 = \mu_1\mu_2\mu_3 \qquad r_3 = \nu_1\nu_2\nu_3.$$

Define the transformation

$$z = Kx.$$

Since $\det \mathbf{K} \neq 0$, $\mathbf{K}^{-1}$ exists and (1) transforms into

$$\dot{z} = \begin{cases} \mathbf{KA}_1\mathbf{K}^{-1}z + \mathbf{K}b & z_1 \geqslant 1 \\ \mathbf{KA}_0\mathbf{K}^{-1}z & -1 \leqslant z_1 \leqslant 1 \\ \mathbf{KA}_{-1}\mathbf{K}^{-1}z + \mathbf{K}c & z_1 \leqslant -1 \end{cases} \tag{4}$$

where

$$\mathbf{KA}_0\mathbf{K}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p_3 & -p_2 & p_1 \end{bmatrix}$$
$$\mathbf{K}b = \begin{bmatrix} p_1 - q_1 \\ q_2 - p_2 + p_1(p_1 - q_1) \\ p_3 - q_3 - p_2(p_1 - q_1) + p_1[q_2 - p_2 + p_1(p_1 - q_1)] \end{bmatrix} \equiv \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \end{bmatrix}$$
$$\mathbf{K}c = \begin{bmatrix} r_1 - p_1 \\ p_2 - r_2 + p_1(r_1 - p_1) \\ r_3 - p_3 - p_2(r_1 - p_1) + p_1[p_2 - r_2 + p_1(r_1 - p_1)] \end{bmatrix} \equiv \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \tilde{c}_3 \end{bmatrix} \tag{5}$$
$$\mathbf{KA}_1\mathbf{K}^{-1} = \begin{bmatrix} -\tilde{b}_1 & 1 & 0 \\ -\tilde{b}_2 & 0 & 1 \\ -\tilde{b}_3 + p_3 & -p_2 & p_1 \end{bmatrix}$$
$$\mathbf{KA}_{-1}\mathbf{K}^{-1} = \begin{bmatrix} \tilde{c}_1 & 1 & 0 \\ \tilde{c}_2 & 0 & 1 \\ \tilde{c}_3 + p_3 & -p_2 & p_1 \end{bmatrix}.$$

Assume that f, $g \in C \backslash C_0$ have the same eigenvalues. Then, both systems can be transformed into (4). As a consequence, if f and g have chaotic attractors A_f and A_g respectively, then they are equivalent: the homeomorphism h is given by the matrix $\mathbf{K}_f\mathbf{K}_g^{-1}$ (or $\mathbf{K}_f^{-1}\mathbf{K}_g$) ($\mathbf{K}_f$ denotes the matrix $\mathbf{K}$ for system f, and $\mathbf{K}_g$ denotes the matrix $\mathbf{K}$ for g). □

In fact, with proposition 1 we have proved much more; namely, that the dynamics of every system in $C \backslash C_0$ can be mapped into the system (4). Thus, the separate analysis of every 15-parameter system in $C \backslash C_0$ is unnecessary: it is enough to make the analysis of the nine-parameter system (4) only.

Note that dynamical systems considered in [2] are elements in $C \backslash C_0$.

The attempt to identify the equivalence classes of the nine-parameter family will be presented in [3].

Bidirectional coupling. Let us consider the following dynamical system:

$$\left.\begin{aligned}\dot{x} &= f(x) + \lambda(y - x)\\ \dot{y} &= g(y) + \mu(x - y)\end{aligned}\right\} \tag{6}$$

where λ, μ are real non-negative parameters. Let us assume that the dynamical systems $\dot{x} = f(x)$, $\dot{y} = g(y)$ and (6) have chaotic attractors A_f, A_g and A respectively. Denote the projection of A on the subspace $x = (x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{R}^3$ by A_x, and on the subspace $y = (y_1, y_2, y_3)^{\mathrm{T}} \in \mathbb{R}^3$ by A_y.

Proposition 2. (i) If $f = g$ and $|x(t = 0) - y(t = 0)|$ is sufficiently small, then there exists a value of $k \equiv \lambda + \mu$, say k_*, such that for $k > k_*$, A_x is equivalent to A_y.

(ii) If $f \neq g$ and $\lambda = \infty$, then A_x is equivalent to A_g.

(ii) If $f \neq g$ and $\mu = \infty$, then A_y is equivalent to A_f.

Proof. For simplicity we present a proof for the piecewise-linear systems case; a general proof will be given in [3].

(i) First note that the inequalities

$$|a_{jj}| > \sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}| \qquad j = 1, \ldots, n$$

are sufficient for the stability of a matrix $[a_{ij}]$ with negative diagonal elements. Denote $u = x - y$, so that from (6) we have

$$u = \{-(\lambda + \mu)\mathbf{E} + \mathbf{D}f|_{u=0}\}u + \mathrm{O}(x, y) \equiv \mathbf{A}u + \mathrm{O}(x, y)$$

where $\mathbf{D}f$ is the Jacobian matrix of f, $\mathbf{E}$ is the unit matrix and $\mathrm{O}(x, y)$ represents the higher-order terms. It is obvious that one can find k such that matrix $\mathbf{A} = [a_{ij}]$ is stable, that is $u = 0$ is asymptotically stable, and $x(t)$ approaches $y(t)$ as $t \to \infty$. Hence, A_x is equivalent to A_y (the homeomorphism h: $\mathbb{R}^3 \to \mathbb{R}^3$ is the identity).

(ii) Equation (6) can be rewritten as

$$\left.\begin{aligned}\varepsilon\dot{x} &= \varepsilon f(x) + (y - x)\\ \dot{y} &= g(y) + \mu(x - y)\end{aligned}\right\} \tag{7}$$

where $\varepsilon = 1/\lambda$. If $\varepsilon = 0$, the last equation is equivalent to

$$\left.\begin{aligned}x &= y\\ \dot{y} &= g(y)\end{aligned}\right\}.$$

Thus, A_x is equivalent to A_g (again, the homeomorphism h: $\mathbb{R}^3 \to \mathbb{R}^3$ is the identity). The proof of (iii) is similar. □

The second and the third part of proposition 2 can be improved in the following way.

Proposition 3. For sufficiently small $|x(t = 0)| + |y(t = 0)|$ and ε, there exists t_0 such that $x(t)$ converges uniformly to $y(t)$ as $\varepsilon \to 0^+$ on all subsets of $t_0 < t < \infty$.

Proof. The proof is similar to the proof of theorem 2 in [4, 7]. □

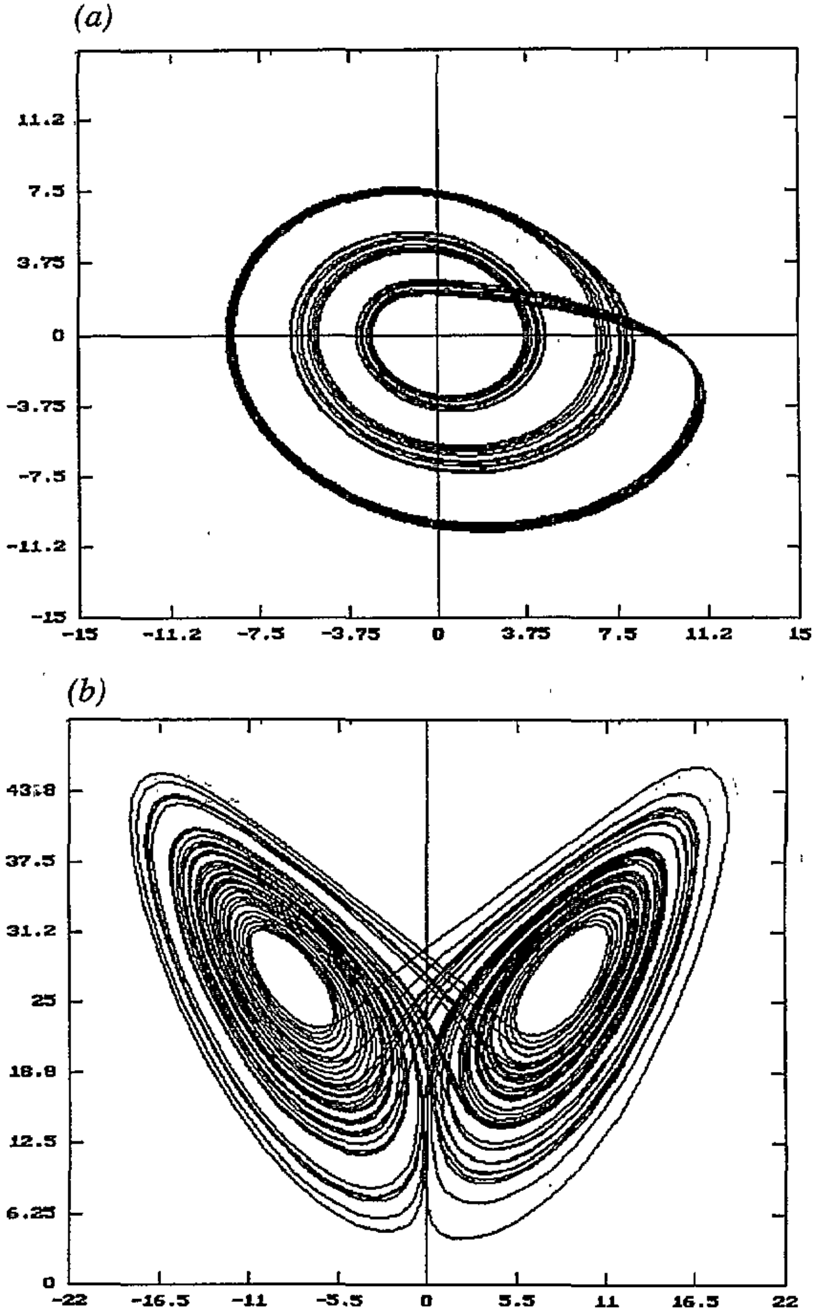

Figure 1. (*a*) Rossler and (*b*) Lorenz attractor.

We shall now consider in more detail the case $f \neq g$ and finite, but with 'large enough' λ. For given dynamical systems we can show numerically that A_x is equivalent to A_y. Assume that the vector field f is that of a Rossler system and g is that of a Lorenz system: $f = (-(x_2 + x_3), x_1 + 0.2x_2,\ 0.2 + x_3(x_1 - 5.57))$, and $g =$

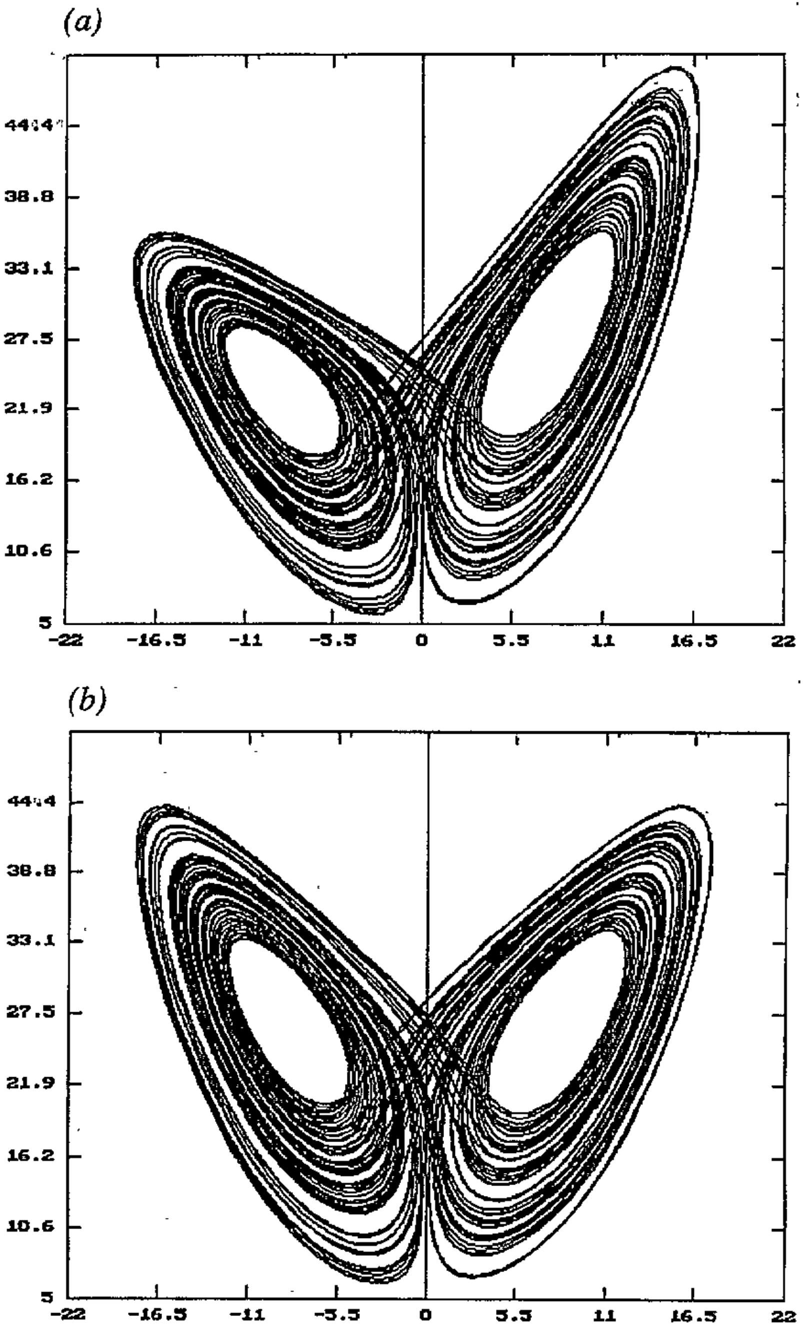

Figure 2. The projection of the chaotic attractor A on the subspace (a) $x = (x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{R}^3$ and (b) on the subspace $y = (y_1, y_2, y_3)^{\mathrm{T}} \in \mathbb{R}^3$.

$(10(-y_1 + y_2), 28y_1 - y_2 - y_1y_3, y_1y_2 - 2.666y_3)$. Figures 1 and 2 show the attractors A_x and A_y for $\lambda = \mu = 0$ and $\lambda = 100$, $\mu = 1$, respectively. Both attractors in figure 2 are similar. Our analysis of the equivalence of these two attractors is based on the fact

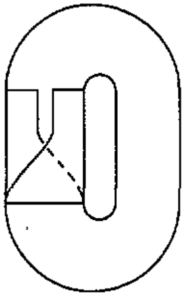

Figure 3. Lorenz template.

that a dense set of unstable periodic orbits are embedded within the chaotic attractor. The topological structure of the flow can be visualized by constructing a template or knot-holder. In its construction we use the approach of Mindlin *et al* [5]: the template is constructed from the attractor by extracting only the lowest periodic orbits embedded in the attractor. The result is shown in figure 3: attractors A_x and A_y have the same template, namely the well known Lorenz template [6]. The template is a tool for computing the topological properties of the periodic orbits embedded in the attractor. Hence, periodic orbits embedded in both attractors A_x and A_y *have the same topological properties.* As a consequence, for two knots K_1 and K_2 of the same type, $K_1 \subset A_x$ and $K_2 \subset A_y$, there exists a homeomorphism of $\mathbb{R}^3$ onto itself which maps K_1 onto K_2. Thus, we conjecture that A_x and A_y are equivalent. Additional evidence that attractors A_k and A_g are equivalent can be given by considering equations (7). If ε is 'small enough', it follows $x \approx y$, that is, the considered chaotic attractors are equivalent.

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